A multivariate model for financial indexes and an algorithm for detection of jumps in the volatility

M. Bonino M. Camelia P.Pigato*

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Abstract

We study a bivariate mean reverting stochastic volatility model, finding an explicit expression for the decay of cross-asset correlations over time. We compare our result with the empirical time series of the Dow Jones Industrial Average and the Financial Times Stock Exchange 100 in the period 1984-2013, finding an excellent agreement. The main features of the model consist in the jumps in the volatilities and a nonlinear mean reversion. Based on these features, we propose an algorithm for the detection of jumps in the volatility.

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1 Introduction

In the last two decades a number of researchers has shown an increasing interest in the field of economics and finance and their links with statistical mechanics. Many interesting phenomena arise when looking at financial data with mathematical tools coming from statistical physics, this being motivated by the fact that a financial market is somehow analogous to a physical "complex system", being the reflexion of the interactions of a huge number of agents. What we are looking at is not the behavior of the single agent, but some macroscopic quantity that we consider important. This new viewpoint has led to the discovery of some striking empirical properties, detected in various types of financial markets, considered now as *stylized facts* of these markets. Examples of such facts are scaling properties, auto-similarity, properties of the volatility profile, such as peaks and clustering, and long range dependence.

In this paper we deal with the two last phenomena, but the point of view we adopt comes from mathematical finance more than statistical mechanics. We do not look at the microscopic behavior, but directly at the macroscopic quantities mentioned above. We can suppose that the large-scale phenomena under study have their origin in some small-scale interactions, but what we try to model here is just the macroscopic world. For this purpose we put ourselves in the framework of continuous-time stochastic volatility models. More precisely, the market models that will be used in this work are mean reverting stochastic volatility models, with a volatility driven by a jump process. This means that our process

*Corresponding author

for the detrended log-price evolves through $dX_t = v_t dB_t$, where B is a Brownian motion and the volatility v is the square root of the stationary solution of a SDE of the following form:

$$dv_t^2 = rev(v_t^2)dt + dL_t. \tag{1}$$

The function rev, what we call "mean reversion", has the role of pushing the volatility back to a certain equilibrium value, whereas $L = (L_t)_t$ is a process which models the noise in the volatility, and it is often taken as a pure jump process (see [9], [14], [2]). If v is an independent process with paths in $L^2_{loc}(\mathbb{R})$ a.s., the process X can be seen as a random time-change of Brownian motion: $X_t = W_{I(t)}$, where $I(t) = \int_0^t v_s^2 ds$. An example of such process is the model presented in [1], that will be the main focus in this article. The introduction of this model is justified by its capability to account a number of the stylized facts mentioned above, namely: the crossover in the log-return distribution from a power-law to a Gaussian behavior, the slow decay in the volatility autocorrelation, the diffusive scaling and the multiscaling of moments, while keeping a simple formulation and an explicit dependence on the parameters. We will push further the analysis of such a model, considering the two following issues: an algorithm for the detection of shocks in the market (peaks in the volatility profile) and a study of a bivariate version of the model. These two aspects are linked by the fact that the cross-correlation between two indexes is in fact the correlation between the two time changes, and these are highly dependent on the jumps of the volatility process.

In section 2 we present the model for time series introduced in [1], briefly enunciating the results proved there. We propose an algorithm for the detection of jumps in the volatility. The problem of finding shocks in financial time series is a classical one. For example, GARCH models (Generalized Autoregressive Conditional Heteroskedasticity, [3]) are widely used, but in practice "volatility seems to behave more like a jump process, where it fluctuates around some value for an extended period of time, before undergoing an abrupt change, after which if fluctuates around a new value" (see [20]). To adress this issue, regime-switching GARCH models have been developed (see [10], [11]), but they can be hard to implement. Therefore, a more common approach is to use an approximate procedure, the so-called ICSS-GARCH algorithm, introduced in [12]. This algorithm is similar to the algorithm that we present because they both use squared returns to detect volatility shocks. However, the ICSS-GARCH algorithm works well under the assumption that the returns are normally distributed. Our algorithm, on the contrary, does not need any particular assumption on the distribution of the returns, but it is simply based on geometrical considerations.

We use our algorithm on the empirical time series of the Dow Jones Industrial Average (DJIA) and the Financial Times Stock Exchange (FTSE) 100, from 1984 to 2013. Some heuristic considerations on the output of the algorithm confirm its validity in the detection of jumps. We find that the majority of the peaks of the volatility estimated by the algorithm are shared by the two indexes. This is a motivation to consider two processes of shocks with a common part.

In section 3 we develop a bivariate version of the model. Motivated by the fact that most of the shocks are common to the two markets, we model the joint process of shocks through correlated Poisson point processes. This is a main ingredient in our modeling since the long range dependence heavily relies, through the volatility process, on the shock times.

Indeed, defining

$$dX_t = v_t^X dB_t^X, \qquad d(v_t^X)^2 = rev((v_t^X)^2)dt + dL_t^X,$$
$$dY_t = v_t^Y dB_t^Y, \qquad d(v_t^Y)^2 = rev((v_t^Y)^2)dt + dL_t^Y,$$

it is easy to show under very weak hypothesis on the volatility $(v_t^X, v_t^Y)_t$ that

$$\lim_{h\downarrow 0} corr(|X_h - X_0|, |Y_{t+h} - Y_t|) = corr(v_0^X, v_t^Y).$$

If the volatilities are of the precise form considered in [1], explicit computations are possible and the evolution of (v^X, v^Y) depends just on the jumps of L^X and L^Y. The correlation of both increments and absolute increments of two assets at a certain time has been widely studied, especially because of its direct link with systemic risk and portfolio management (see for instance [8], [5]), but here we deal with something more peculiar. We consider the cross-correlation of absolute increments at different times, and compute how this correlation decays as the time difference increases. This issue has been addressed by Podobnik et al. in [18], where they analyze the Dow Jones industrial and the S&P500 indexes, and in [19], [21], where long range cross-correlations between magnitudes are found in a number of studies including nanodevices, atmospheric geophysics, seismology and finance. In our framework we find this explicit formula for the decay of cross-asset correlations between absolute returns depending on the time lag, analogous to the formula for the decay of autocorrelations (see Corollary 1):

$$\begin{aligned}\lim_{h\downarrow 0} \frac{Cov(|X_h - X_0|, |Y_{t+h} - Y_t|)}{h} =& \frac{4}{\pi}\sigma^X\sigma^Y\sqrt{D^X D^Y}\left(\lambda^X\right)^{1/2-D^X}\left(\lambda^Y\right)^{1/2-D^Y}\times \\ & Cov\left((S^X)^{D^X-1/2}, (\lambda^Y t + S^Y)^{D^Y-1/2}\right)e^{-\lambda^Y t}\end{aligned}$$

The quantities involved are constant parameters of the volatilities v^X and v^Y, except from S^X and S^Y which are correlated exponential variables coming from the jump process $L = (L^X, L^Y)$. We apply this result to the two empirical time series, finding an excellent agreement between predictions of the model and empirical findings. In particular we find that from both modeling and empirical data the decay of autocorrelations and cross-correlations is almost coincident, and in particular it decays slowly over time, confirming this is a long-memory processes. On the other hand, we empirically find a non-significant cross-correlation between returns of FTSE and DJIA, even for very small time lags (see Fig. 1), and this is consistent with the model as well. This is not surprising, since for both

FIGURE 1: Decay of cross-correlations

indexes there are no long-range autocorrelations of returns, and this is easily seen to be consistent with our model. In contrast, as already said, the decay of cross-correlation of absolute returns is very slow.

The fact that our model displays a behavior that is completely analogous to real data in all of these aspects, even the most subtle, is as an interesting validation of our model. We mention that the satistical analysis performed by Podobnik et al. in [18],[21] lead to results analogous to ours, concerning also the similarity in the decay of autocorrelations and

cross-asset correlations. Their conclusion is that "Once the volatility (risk) is transmitted across the world, the risk lasts a long time".

We conclude these considerations saying that we believe this model flexible enough to provide a satisfying fit also of equal time correlations, of both returns and absolute returns, since we are able to correlate volatilities and Brownian motions with really explicit dependence on the parameters. This problem is being considered at the moment and will be included in a future work.

In section 4 we prove formally some results justifying the convergence of the algorithm for the detection of shocks in the volatility. We stick to this precise model for the linearity of the exposition but the same proof would give an analogous result for a wider class of volatilities solving (1) (see [16]).

Section 5 contains proofs of the results related with the bivariate model.

2 Detection of jumps in the univariate model

In this section we describe the model and state properties and results related to stylized facts. We then propose some considerations on quadratic variation, and develop an algorithm for the detection of big shocks in the market.

2.1 Definition of the model

Given three real numbers $D \in (0, 1/2]$, $\lambda \in (0, \infty)$, $\sigma \in (0, \infty)$, the model is defined upon two sources of randomness:

- a Brownian motion $W = (W_t)_{t \geq 0}$;
- a Poisson point process $\mathcal{T} = (\tau_n)_{n \in \mathbb{Z}}$ of rate λ on $\mathbb{R}$.

We suppose W and $\mathcal{T}$ independent. By convention we label the points of $\mathcal{T}$ so that $\tau_0 < 0 < \tau_1$. For $t \geq 0$, we define

$$i(t) := \sup\{n \geq 0 : \tau_n \leq t\} = \#\{\mathcal{T} \cap (0, t]\}.$$

$i(t)$ is the number of positive times in the Poisson process before t, so that $\tau_{i(t)}$ is the location of the last point in $\mathcal{T}$ before t. We introduce the process $I = (I_t)_{t \geq 0}$ defining

$$I_t = \sigma^2 \left[(t - \tau_{i(t)})^{2D} + \sum_{k=1}^{i(t)} (\tau_k - \tau_{k-1})^{2D} - (-\tau_0)^{2D} \right] \tag{2}$$

where we agree that the sum in the right hand side is 0 if $i(t) = 0$. Now we define the process which is the model for the detrended log price as

$$X_t = W_{I(t)}. \tag{3}$$

Observe that I is a strictly increasing process with absolutely continuous paths, and it is independent of the Brownian motion W. Thus this model may be viewed as an independent random time change of a Brownian motion.

We shortly give a motivation for this definition. Remark that for $D = 1/2$ the model reduces to Black & Scholes with volatility σ. For $D < 1/2$, the introduction of a time

inhomogeneity $t \to t^{2D}$ at times τ_n is meant to represent the *trading time* of a financial time series, where at "random" times there are shocks in the market, modeled by our Poisson point process. The reaction of the market is an acceleration of the dynamics immediately after the shock, and a gradual slowing down at later times, until a new shock accelerates the dynamics again. This behavior is due to the shape of the function $t \to t^{2D}$, $D \in (0, 1/2]$, which is steep for t close to 0 and bends down for increasing t.

FIGURE 2: Time inhomogeneity

The definition of the model as a time changed Brownian Motion implies that we can equivalently express it as a a stochastic volatility model, where the volatility is

$$v_t = \sqrt{I'(t)} = \sqrt{2D}\sigma(t - \tau_{i(t)})^{D-1/2},$$

and the evolution of X is given by $dX_t = v_t dB_t$. To write the model as solution of a two-dimensional stochastic differential equation we can define the volatility as the the stationary solution of

$$d(v_t^2) = -\alpha(v_t^2)^\gamma dt + \infty \, di(t),$$

where the constants are

$$\gamma = 2 + \frac{2D}{1-2D} > 2, \quad \alpha = \frac{1-2D}{(2D)^{1/(1-2D)}} \frac{1}{\sigma^{2/(1-2D)}}.$$

This process is well defined, since after the infinite jumps the super-linear drift term instantaneously produces a finite pathwise solution. We refer to [7] for the details of the correspondence between time change and stochastic volatility in this framework, for a wider class of stochastic volatility models.

Remark 1. In the most general version of this model σ is not constant. A sequence of random variables $(\sigma_n)_{n\in\mathbb{N}}$ is simulated, and each of them is associated to the corresponding jump. All the results presented here are still valid in this case, with a slightly more complicated formulation. We have decided to assume σ constant in this work, since calibration on data coming from financial time series leads in any case to this type of choice. Moreover, this assumption does not significantly alter the model, allowing on the other hand more agile statements for hypothesis and theorems, in particular for what concerns Corollary 1 and Theorem 5.

2.2 Main properties

We enounce briefly some properties of the process X. For proofs, more detailed statements and some additional considerations we refer to [1].

Proposition 1 (Basic Properties). *Let X be the process defined in* (3). *The following assertions are satisfied:*

1. *X has stationary increments.*
2. *X is a zero-mean, continuous, square-integrable martingale, with quadratic variation $\langle X \rangle_t = I_t$.*
3. *The distributions of the increments of X is ergodic.*
4. *$E(|X_t|^q) < \infty$ for some (and hence any) $t > 0$, $q \in [0, \infty)$.*

We are now ready to state some results, important because they establish a link between our model and the stylized fact presented in the introduction. In the limit for $h \downarrow 0$, the process X defined in (3) and is consistent with important facts empirically detected in many (financial) real time series, namely: diffusive scaling of returns, multiscaling of moments, slow decay of volatility autocorrelation.

Theorem 1 (Diffusive scaling). *We have that in distribution, for $h \downarrow 0$,*

$$\frac{X_{t+h} - X_t}{\sqrt{h}} \longrightarrow f(x)dx \tag{4}$$

where f is a mixture of centered Gaussian densities

$$f(x) = \int_0^\infty dt\, \lambda e^{-\lambda t} \frac{t^{1/2-D}}{\sigma\sqrt{4D\pi}} exp\left(-\frac{t^{1-2D}x^2}{4D\sigma^2}\right).$$

From this result, it follows that

$$\mathbb{E}_f(|x|^q) = \infty \Leftrightarrow q \geq q^* := (1/2 - D)^{-1}.$$

The function f, which describes the asymptotic law, for $h \downarrow 0$, of $\frac{X_{t+h}-X_t}{\sqrt{h}}$, has a different tail behavior from the density of $X_{t+h} - X_t$, for fixed h (cf. Proposition 1 point 4).

This feature of f is linked to another property of our model: the multiscaling of moments. Let us define the $q-th$ moment of the log returns, at time scale h:

$$m_q(h) := \mathbb{E}\left(|X_{t+h} - X_t|^q\right) = \mathbb{E}\left(|X_h|^q\right)$$

the last equality holding for the stationarity of increments. Because of the diffusive scaling properties (4), we would expect $m_q(h)$ to approximate in some sense $h^{\frac{q}{2}} \int x^q f(x)dx = C_q\, h^{\frac{q}{2}}$, for $h \downarrow 0$. This is actually true for $q < q^*$, that is, for q such that the $q-th$ moment of the limit distribution is finite. For $q \geq q^*$, the $q-th$ moment of the limit distribution is not finite, and it turns out that a faster scaling holds, namely $m_q(h) \approx h^{Dq+1}$. This transition in the scaling of $m_q(h)$ is known as multiscaling of moments, a property empirically detected in many time series, in particular in financial series. The following theorem states that for this model the multiscaling exponent is a piecewise linear function of q. In [7] the problem of multiscaling in more general stochastic volatility models is considered, finding that an analogous behavior is common to a much wider class of models.

Theorem 2 (Multiscaling of moments)**.** *For $q > 0$ the $q - th$ moment of log returns $m_q(h)$ has the following asymptotic behavior as $h \downarrow 0$:*

$$m_q(h) \sim \begin{cases} C_q h^{\frac{q}{2}}, & if\ q < q^* \\ C_q h^{\frac{q}{2}} \log\left(\frac{1}{h}\right), & if\ q = q^* \\ C_q h^{Dq+1}, & if\ q > q^* \end{cases}$$

for some constants $C_q \in (0, \infty)$ (whose explicit expression can be found in [1]). As a consequence, the scaling exponent $A(q)$ is

$$A(q) = \lim_{h \downarrow 0} \frac{\log m_q(h)}{\log h} = \begin{cases} \frac{q}{2} & if\ q \leq q^* \\ Dq + 1 & if\ q \geq q^* \end{cases}$$

We now state a result concerning the volatility autocorrelation of the process X, that is the correlations of absolute values of returns at a given time distance. Recall that the correlation coefficient of two random variables X and Y is

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}.$$

For the process X, introduce $\xi = (\xi_t)_{t \geq 0}$, the process of absolute values of increments, for h fixed: $\xi_t = |X_{t+h} - X_t|$. Then the volatility autocorrelation of X is

$$\rho(t - s) = \rho(\xi_s, \xi_t) = \frac{Cov(\xi_s, \xi_t)}{\sqrt{Var(\xi_s)Var(\xi_t)}}$$

Indeed, being the process stationary, the quantity we have defined above depends just on the time difference $t - s$. Let's state our result, concerning as above the asymptotic behavior as $h \downarrow 0$.

Theorem 3 (Volatility autocorrelaton)**.** *For $t \geq 0$,*

$$\lim_{h \downarrow 0} \rho(t) = \frac{2}{\pi} \frac{Cov\left(S^{D-1/2}, (\lambda t + S)^{D-1/2}\right)}{Var(|N| S^{D-1/2})} e^{-\lambda t}$$

where S is an exponential variable with parameter 1, N is a standard normal variable and they are mutually independent.

This theorem is actually a special case of Corollary 1. This theorem shows that the decay of volatility autocorrelation is between polynomial and exponential for $t = O(1/\lambda)$, exponential for $t >> 1/\lambda$.

2.3 Heuristics on quadratic variation

As mentioned in [1], the instants in $\mathcal{T}$ are "the ephocs at which big shocks arrive in the market, making the volatility jump to infinity". The aim of the following considerations is an algorithm for finding the location of relevant big jumps in the volatility, proposed for the first time in [4].

On one hand we know that the quadratic variation of X is given by I (Proposition 1), i.e.

$$\langle X \rangle_t = I_t.$$

Therefore, since we know that the quadratic variation of X_t is the limit in probability of the squared increments on shrinking partitions, it seem natural to estimate I by evaluating the squared increments of a dense sampling of X.

On the other hand, the process I is piecewise-concave; in fact, we recall that such process is defined by

$$I_t = \sigma^2 \left[(t - \tau_{i(t)})^{2D} + \sum_{k=1}^{i(t)} (\tau_k - \tau_{k-1})^{2D} - (-\tau_0)^{2D} \right]$$

It is clear that between two consecutive shock times the process is concave. Therefore, if we are at time T and we consider the backward difference quotient defined by

$$Q_T(t) := \frac{I_T - I_{T-t}}{t}$$

this quantity is, conditional on $\mathcal{T}$, increasing up to the last shock time before T, therefore it has a local maximum in $t = T - \tau_{i(T)}$. Moreover, the derivative of I_t is very big after a shock but it quickly decays over time. Because of that, we expect that $Q(s) < Q(T - \tau_{i(T)})$ if $s \in (T - \tau_{i(T)}, T - \tau_{i(T)} - L)$, for some $L > 0$. We propose here an algorithm based on the following idea: if we choose a $M > 0$ such that $T - M < \tau_{i(T)} < T$ and $T - M$ is "closer to $\tau_{i(T)}$ than to $\tau_{i(T)-1}$", then the global maximum of $Q(t)$ in the interval $(T - M, T)$ should be attained in $t = T - \tau_{i(T)}$.

In view of these two observations, we introduce the following estimator

$$V_T(k) := \frac{1}{k} \sum_{i=1}^{k} (X_{T-i+1} - X_{T-i})^2$$

In our data analysis, the estimator is an average of daily squared increments of X_t (the densest sampling we could get), so it should be a good estimate of the quadratic variation of X. Moreover, it has the following property

$$\begin{aligned}
\mathbb{E}\{V_T(k)|\mathcal{T}\} &= \frac{1}{k} \sum_{i=1}^{k} \mathbb{E}\left\{ (X_{T-i+1} - X_{T-i})^2 |\mathcal{T} \right\} \\
&= \frac{1}{k} \sum_{i=1}^{k} \mathbb{E}\left\{ \left(W_{I_{T-i+1}} - W_{I_{T-i}}\right)^2 |\mathcal{T} \right\} \\
&= \frac{1}{k} \sum_{i=1}^{k} (I_T - I_{T-k}) \\
&= Q_T(k)
\end{aligned}$$

Therefore, since we can't observe Q directly, we can use V_T as an approximation of it and try to apply the idea of the algorithm outlined above by estimating V_T on historical data. In Section 4 we give an accurate mathematical explanation of these heuristics, motivating the algorithm presented below.

2.4 The Algorithm

We outline now the empirical use of the algorithm. First of all we introduce some notation. The financial index time series will be denoted by $(s_i)_{0\le i\le N}$, whereas the detrended logarithmic time series will be indicated by $(x_i)_{250\le i\le N}$, where

$$x_i := \log(s_i) - \bar{d}(i)$$

and $\bar{d}(i) := \frac{1}{250}\sum_{k=i-250}^{i-1}\log(s_i)$; we observe that it is not possible to define x_i for $i < 250$. We moreover define $(y(i))_{0\le i\le N}$ to be the corresponding series of the trading dates. We also introduce the empirical estimate of V_N as

$$\widehat{V_N}(k) := \frac{1}{k}\sum_{i=1}^{k}(x_{N-i+1} - x_{N-i})^2$$

Now, suppose that we want to know when the last shock time in the time series occurred. We recall that the idea is to choose an appropriate integer M such that $0 < M \le N$ and see where the sequence $(\widehat{V_N}(k))_{N-M\le k\le N}$ attains its maximum. This leads us to introduce the following definition.

Definition 1. Let $(s_i), (x_i), (y_i), N, M$ be as above; given as integer $\tilde{N}$ such that $M \le \tilde{N} \le N$, we define

$$\widehat{k}(\tilde{N}, M) := \mathrm{argmax}_{\tilde{N}-M\le k\le\tilde{N}} \quad \frac{1}{k}\sum_{i=1}^{k}\left(x_{\tilde{N}-i+1} - x_{\tilde{N}-i}\right)^2.$$

This quantity is an estimate of the distance of the last shock time before $y_{\tilde{N}}$ from $\tilde{N}$. Using this we define also

$$\widehat{i}(\tilde{N}, M) := \tilde{N} - \widehat{k}(\tilde{N}, M) + 1,$$

our estimate of the index of the last shock time estimate, and consequently our estimate of the last shock time before $y_{\tilde{N}}$ is

$$\widehat{\tau}(\tilde{N}, M) := y\left(\widehat{i}(\tilde{N}, M)\right).$$

It is worth compare briefly our algorithm with the so called ICSS-GARCH algorithm. Following [20], we can describe the ICSS-GARCH algorithm as follows. Given a series of financial returns $r_1, \dots, r_n$, with mean 0 we define the cumulative sum of squares $C_k = \sum_{i=1}^{k} r_i^2$ and let

$$D_k = \frac{C_k}{C_n} - \frac{k}{n}, \qquad 1 \le k \le n, \qquad D_0 = D_n = 0$$

The idea is that if the sequence $r_1, \dots, r_n$ has constant variance, then the sequence $D_1, \dots, D_n$ should oscillate around 0. However, if there is a shock in the variance, the sequence should exhibit extreme behavior around that point.

We remark that both algorithms use squared returns to detect volatility shocks. However, the ICSS-GARCH algorithm works well under the assumption that the returns are normally distributed, but not with heavy-tailed distributions, as proven in [20]. Our algorithm, on the

contrary, does not need any particular assumption on the distribution of the returns, but it is simply based on geometrical considerations. In fact it exploits the particular characteristics of a piecewise-concave Brownian motion time change to locate shocks. We point out the assumption of a piecewise-concave Brownian motion time change is very natural in the context of stochastic volatility models. In fact, to reproduce jumps in the volatility, one has to introduce a process that makes the volatility dramatically increase when a shock occurs, and then slowly decay over time. To reproduce such a behavior, it seems natural to introduce a piecewise-concave time change. Furthermore, we remark that this algorithm does not work just with the model that presented here. For instance it is possible to prove that it works with any model where the detrended log-price is given by W_{J_t}, where W is a Brownian motion and J_t is a time change such that

$$J_t = g(t - \tau_{i(t)}) + \sum_{k=1}^{i(t)} g(\tau_k - \tau_{k-1}),$$

with $\{\tau_i\}_{i\in\mathbb{Z}}$ and $i(t)$ as in the model in [1] and $g : [0, +\infty) \to [0, +\infty)$ is concave and satisfies $g(0) = 0$, $\lim_{h\to 0^+} g(h)/h = +\infty$. [1]

Recall that for the empirical discussion outlined here, we decided to use the DJIA Index and FTSE Index, from April 2nd, 1984 to July 6th, 2013, so that $N = 7368$. A similar data analysis has been done on the Standard & Poor's 500 Index, from January 3rd, 1950 to July 23th, 2013, finding analogous result and confirming the validity of the method we present here on aggregate indexes. All the calculations and pictures presented here have been obtained using the software MatLab [15]. An example of the empirical procedure to estimate the last shock time is given in Figure 3.

FIGURE 3: Plot of the quantity $\widehat{V_{\tilde{N}}}(k)$ for $k = 1, \ldots, 2000$ ($M = 2000$). $\tilde{N}$ has been choosen so that $y_{\tilde{N}}$ is the 10th of May 2011. The peak corresponds to the the 15th of September 2008, the day of the Lehman Brothers bank bankruptcy.

However, we are not sure whether the choice of M that we made is good or not. Therefore, to confirm that the shock time estimate is good, we may repeat the estimate approaching the shock time, for example dropping the last observation, or dropping a particular number of the last observations. Then we can repeat this procedure many times and if we see that the last shock time estimate is confirmed, then we have a clear indication of the presence of a shock there (see Figure 4–(a)). We remark that when more than one shock is present on the time interval we consider, the most recent is always found as the maximum peak of $\hat{V}$ if we take $y_{\tilde{N}}$ close enough to it. When we get further, the chosen peak is not necessarily the

[1] In the model presented above, $g(h) = h^{2D}$

most recent, as we can see in Figure 4–(b). This is exactly what we expect from Theorem 5.

FIGURE 4: Plot of the quantities $\widehat{V_{\tilde{N}}}(k)$ for $k = 1, \dots, 2000$ ($M = 2000$), for the DJIA. In each figure we shift $\tilde{N}$ 4 times of 20 working days. In (a) $\tilde{N}$ has been chosen so that $y_{\tilde{N}}$ is the 10/05/11(red), the 11/04/11(yellow), the 14/03/11(green) and the 11/02/11(blue). The four maxima are all located the 15/09/08, the day of the Lehman Brothers bank bankruptcy, confirming the presence of a shock there. In (b) $\tilde{N}$ has been chosen so that $y_{\tilde{N}}$ is the 27/02/12(red), the 27/01/12(yellow), the 28/12/11(green) and the 29/11/11(blue). We can see that when $y_{\tilde{N}}$ is close to the 05/08/11 (European sovereign debt crisis), this date corresponds to the maximum of $\hat{V}$, whereas when we move further the maximum is again on the 15/09/08.

(a) (b)

We can now apply the algorithm to try to locate all the past shocks in a given financial index time series. To do so, we simply calculate the quantity $\widehat{i}(\tilde{N}, M)$ for $\tilde{N} = N, \dots, M$. We introduce the following sequence.

Definition 2. Given the quantities defined in definition 1, we introduce the past shock time sequence as

$$\widehat{h}((x_i)_{250 \leq i \leq N}, M) := \left(\widehat{i}(\tilde{N}, M)\right)_{M \leq \tilde{N} \leq N}$$

However, we slightly tweak the procedure in order to get a clearer result. When calculating $\widehat{k}(\tilde{N}, M)$ we ignore the last 20 elements of the sum, in order words, instead of calculating $\widehat{k}(\tilde{N}, M)$ as the argmax for $\tilde{N} - M \leq k \leq \tilde{N}$, we drop the last 20 elements of the series. This leads us to recognize shocks that are at least 20 days old, removing the noise due to the excess volatility. We do this because when $\tilde{N}$ is very close to a shock, the procedure becomes unstable since near a shock the volatility is very high, so it is not always clear where the maximum is.

Finally, to get a clear picture of which are the big shocks in the time series, we can plot the number of the occurencies of each element of the sequence $\widehat{h}((x_i)_{250 \leq i \leq N}, M)$. We may choose to consider an element of the sequence of dates a shock if its numbers of occurrences exceeds a certain threshold. Table 1 contains our estimated shock-dates. In Figure 5 you can see the graphical evidence that maxima are concentrated on a small set of days for both FTSE and DJIA, supporting the validity of the method. The choice of the threshold is not completely determined, and we have based it on two criteria. Firstly, the number of estimated shocks should be consistent with the number of expected jumps of the Poisson process (whose rate will be calibrated in section 3). Secondly, we see that in both series it

is possible to find a big interval in $\mathbb{N}$ for which almost no date has a number of occurrences contained in that interval. More explicitly, for the DJIA there are just 3 dates found approximately 50 times, whereas all the others are found more than 80 times or less than 25. Analogously, for the FTSE there are just 2 dates found approximately 50 times, whereas all the others are found more than 80 times or less than 20. It is therefore reasonable to consider true shocks the ones occurred more than 80 times, whereas it is not that clear how to consider the dates with approximately 50 occurrences. In any case, these choices are consistent with the number of expected jumps of the Poisson process. Another issue in the choice of shock dates is the fact that sometimes there are two or more very close dates which are found a considerable number of times. In this case we consider them as related to the same shock. These dates are marked with the word "sparse" in table 1, where we have reported our estimated dates.

FIGURE 5: Shock times; x-axis: increasing time index; y-axis: y(i)=number of times the maximum of $\hat{V}$ is realized at i

(a) FTSE shock times

(b) DJIA shock times

It is natural at this point to wonder if there is a relation between the shocks in the two indexes, and a straightforward experiment is to try to superimpose the two graphics (see Figure 6). What we get is a clear indication that the shock times of the two series are almost coincident, only the magnitude (or evidence) being different and having very few shocks which are present just in one of the two indexes. This is an important hint in the choice of the volatility for the bivariate process, a problem tackled in the following section.

3 The bivariate model

We investigate here the decay of correlation of the absolute returns of a bivariate version $(X, Y) = (X_t, Y_t)_{t\geq 0}$ of the model defined in Section 2, finding a result analogous to Theorem 3. For the two-dimensional time change we chose a structure justified by the findings of subsection 2.4. Proofs are postponed to Section 5. This section is based on [17], where the similar results are proved under weaker assumptions.

3.1 Definition of the bivariate model

The bivariate version of the model must be defined upon the following quantities:

- two Brownian motions $W^X = \left(W_t^X\right)_{t\geq 0}$ and $W^Y = \left(W_t^Y\right)_{t\geq 0}$;

Table 1: Estimated dates of shock times

FTSE	*DJIA*
14/10/87	15/09/87
24/01/89	
26/09/89	11/10/89 (questionable, 53)
	09/01/96
	01/07/96 (questionable, 54)
	13/03/97 (sparse)
08/08/97	
22/10/97 (questionable, 48)	16/10/97
04/08/98	31/07/98
30/12/99	04/01/00
09/03/01	09/03/01
06/09/01	06/09/01
12/06/02	05/07/02
12/05/07 (questionable, 43)	
24/07/07	24/07/07 (questionable, 57)
15/01/08	04/01/08 (sparse)
03/09/08	15/09/08
05/08/11	05/08/11

Figure 6: Common jumps: overlap of Figures 5 (a) and (b)

- two Poisson point processes on $\mathbb{R}$: $\mathcal{T}^X = (\tau_n^X)_{n\in\mathbb{Z}}$ and $\mathcal{T}^Y = (\tau_n^Y)_{n\in\mathbb{Z}}$, of rates respectively λ^X and λ^Y;
- positive constants D^X, D^Y, σ^X and σ^Y.

The tricky point is the definition of the Poisson processes, that we want dependent but different. We introduce $\mathcal{T}^i$, $i = 1, 2, 3$ independent Poisson point processes with intensities λ_i, $i = 1, 2, 3$. Then we define $\mathcal{T}^X = \mathcal{T}^1 \cup \mathcal{T}^2$, $\mathcal{T}^Y = \mathcal{T}^1 \cup \mathcal{T}^3$. These are again Poisson processes, with intensity $\lambda_1 + \lambda_2$ and $\lambda_1 + \lambda_3$, and they are actually mutually dependent if $\mathcal{T}^1$ is non-degenerate.

We want to have a correlation coefficient $\rho \in [-1, 1]$ also between the Brownian motions, and this is a more standard issue in financial modeling. We introduce two independent Brownian motions W^X, $\tilde{W}$, and define

$$W_t^Y = \rho W_t^X + \sqrt{1-\rho^2}\tilde{W}_t.$$

The correlation between W^Y and W^X will play no role in this paper, but the parameter ρ is important for the correlation of the increments of X and Y at the same time, a very important aspect that will be treated in a future work.

We suppose that the two-dim Brownian $W = (W^X, W^Y)$ and the two-dim time change $\mathcal{T} = (\mathcal{T}^X, \mathcal{T}^Y)$ are independent. The requirements of section 2 on the marginal one-dim processes are satisfied and we can define X and Y as

$$X_t = W^X_{I_t^X}, \quad Y_t = W^Y_{I_t^Y}$$

where the random time changes I_t^X and I_t^Y are defined as in (2). This definition is motivated by the fact that very rarely the occurrence of a shock in one of the two indexes does not coincide with a peak in the volatility of the other. So it is reasonable to suppose that a part of the shock process is "common".

3.2 Covariance and correlations of absolute log-returns

For a given time h,

$$\xi_t = |X_{t+h} - X_t|, \quad \eta_t = |Y_{t+h} - Y_t|,$$

are the absolute values of the returns of X and Y at time t. We are interested in the correlations between these two variables, and we start from computing their covariance. In fact, we are now going to state a result on the asymptotic behavior of the covariance of log-returns as the time scale goes to 0.

Theorem 4 (Covariance of absolute log-returns)**.** *Let the process (X, Y) be defined as above. Then, for any $t \geq s \geq 0$, the following holds:*

$$\lim_{h\downarrow 0} \frac{Cov(\xi_s, \eta_t)}{h} = \lim_{h\downarrow 0} \frac{Cov(\xi_0, \eta_{t-s})}{h} =$$
$$\frac{4\,\sigma^X \sigma^Y \sqrt{D^X D^Y}}{\pi} Cov\left((-\tau_0^X)^{D^X-1/2}, (t-s-\tau_0^Y)^{D^Y-1/2}\right) e^{-\lambda^Y(t-s)}$$

Remark 2. Using the definition of $\mathcal{T}^X$ and $\mathcal{T}^Y$ and the properties of Poisson processes it is possible to rewrite this expression as

$$\begin{aligned}\lim_{h\downarrow 0}\frac{Cov(\xi_0,\eta_t)}{h} =& \frac{4}{\pi}\sigma^X\sigma^Y\sqrt{D^X D^Y}\left(\lambda^X\right)^{1/2-D^X}\left(\lambda^Y\right)^{1/2-D^Y}\times\\ & Cov\left((S^X)^{D^X-1/2},(\lambda^Y t+S^Y)^{D^Y-1/2}\right)e^{-\lambda^Y t}\end{aligned}$$

where $S^X=\min\{S^{1,X},S^2\}$ and $S^Y=\min\{S^{1,Y},S^3\}$ are correlated exponential variables of parameter 1, and

$$\begin{aligned}&(\lambda_1+\lambda_2)S^{1,X}=(\lambda_1+\lambda_3)S^{1,Y}\sim\exp\left(\lambda_1\right),\\ &S^2\sim\exp\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right),\quad S^3\sim\exp\left(\frac{\lambda_3}{\lambda_1+\lambda_3}\right)\end{aligned}$$

are mutually independent.

Remark 3. If instead of taking absolute returns we consider simple returns, we find that $\lim_{h\downarrow 0}Cov(X_h-X_0,Y_{t+h}-Y_t)=0$, for any $t>0$. This is why we said in the introduction that our model is consistent with the fact that empirical cross-correlations of returns are not significant even for very small time lags, in analogy with the autocorrelations.

From this theorem we obtain an asymptotic evaluation for correlations between log-returns, when the time scale goes to 0. Recall that the correlation coefficient between ξ_0 and η_t is defined as

$$\rho(\xi_0,\eta_t)=\rho(|X_h|,|Y_{t+h}-Y_t|)=\frac{Cov(\xi_0,\eta_t)}{\sqrt{Var(\xi_0)Var(\eta_t)}}.$$

Corollary 1 (Decay of cross-asset correlations)**.** *For the process (X,Y) defined above, for any $t\geq s\geq 0$, the following expression holds as $h\downarrow 0$:*

$$\lim_{h\downarrow 0}\rho(\xi_0,\eta_t)=\frac{2}{\pi}\frac{Cov\left((S^X)^{D^X-1/2},(\lambda^Y t+S^Y)^{D^Y-1/2}\right)}{\sqrt{Var(|N|S^{D^X-1/2})Var(|N|S^{D^Y-1/2})}}e^{-\lambda^Y t}$$

where with S we denote an exponential variable of parameter 1 and with N a standard normal variable, they are mutually independent and both independent from all the other random variables. S^X and S^Y are defined in Remark 2.

Remark 4. Suppose we are dealing with X and Y produced by the same time change of two different Brownian motions, i.e $I^X=I^Y=:I$, or:

$$D^X=D^Y,\quad \mathcal{T}^X=\mathcal{T}^Y,\quad \sigma^X=\sigma^Y.$$

The expression for the decay of cross-asset correlation becomes in this case

$$\lim_{h\downarrow 0}\rho(\xi_0,\eta_t)=\frac{2}{\pi}\frac{Cov\left(\sigma S^{D-1/2},\sigma(\lambda t+S)^{D-1/2}\right)e^{-\lambda t}}{Var(\sigma|N|S^{D-1/2})},$$

which is exactly the expression for the decay of autocorrelation coefficients (cf. Theorem 3). An analysis of real data suggests that this property is very close to what we see in financial markets.

3.3 Empirical results

We consider again the DJIA Index and FTSE Index, from April 2nd, 1984 to July 6th, 2013. For the data analysis we use the software MatLab [15].What follows is justified by the ergodic properties of the increments X. We start considering the two series separately. We want to assign to the parameters some values such that the predictions of the model are as close as possible to real data. For this purpose, we chose some significant quantities (taking into account interesting features related to stylized facts), and focus on them for the calibration. Here we have considered the multiscaling coefficients C_1 and C_2, the multiscaling exponent $A(q)$, the volatility autocorrelation function $\rho(t)$. The procedure for the calibration is described precisely in [1] for what concerns the one-dimensional model. Here we just outline the basic idea, which is to minimize an L^2 distance between predictions of the model and empirical estimations of these significant quantities. The details of the calibration of the bivariate version can be found in [17]. We find the following estimates for the parameters.

$$\text{FTSE:} \quad \overline{D} \approx 0.16; \quad \overline{\lambda} \approx 0.0019; \quad \overline{\sigma} \approx 0.11.$$
$$\text{DJIA:} \quad \overline{D} \approx 0.14; \quad \overline{\lambda} \approx 0.0014; \quad \overline{\sigma} \approx 0.127.$$

In Figure 7 we show the empirical multiscaling exponent versus the prediction of our model with this parameters. Our estimate for the multiscaling exponent looks smoothed out by the empirical curve. Since a simulation of daily increments of the model yields a graph analogous to the empirical one, this slight inconsistency is likely due to the fact that the theoretical line shows the limit for $h \downarrow 0$, whereas the empirical data come from a daily sample.

Analogously Figure 8 concerns volatility autocorrelation. The decay is between polynomial and exponential, and fits very well empirical data considering the fact that they are quite widespread. We conclude that the agreement is excellent for both multiscaling and volatility autocorrelation.

FIGURE 7: Multiscaling exponent

(a) FTSE

(b) DJIA

We display now the distribution of log returns for our model $p_t(\cdot) = \mathbb{P}(X_t \in \cdot) = \mathbb{P}(X_{n+t} - X_n \in \cdot)$ for $t = 1$ day, and the analogous empirical quantity. We do not have an explicit analytic expression for p_t, but we can easily obtain it numerically. Figure (9) represent the bulks and the integrated tails of the distributions. We see that the agreement is remarkable, given that this curves are a test *a posteriori*, and no parameter has been estimated using these distributions!

Figure 8: Volatility autocorrelation

(a) FTSE log plot

(b) DJIA log plot

(c) FTSE loglog plot

(d) DJIA loglog plot

We have seen in subsection 2.4 that our estimates for jumps in FTSE and DJIA are strictly related. The occurrence of a jump for an index comes very often together with the occurrence of a jump for the other index, with a difference of a few days.

As a consequence, a first idea to try a rough modeling of cross asset correlations is to suppose $\mathcal{T}^f$, jump process for FTSE, and $\mathcal{T}^d$, jump process for DJIA, to be the same process. But if this is true from Remark 4, and from the fact that D and σ are very similar for FTSE and DJIA, we would expect the decay of volatility autocorrelation in the DJIA, the decay of volatility autocorrelation in the FTSE and the decay of cross-asset correlation of absolute returns to display a similar behavior. In fact, this is exactly what happens if we plot these quantities (see Figure 10), in agreement with the empirical findings of [18].

Under this rough hypothesis our estimate for cross-asset correlations is therefore our prediction for the decay of volatility autocorrelation in FTSE or DJIA, or a mean between the two.

We can do better using the bivariate jump process $I = (I^X, I^Y)$ described at the beginning of this section. We just have to estimate the intensities $\lambda_1, \lambda_2, \lambda_3$, subject to the constraints coming from the estimates of the one dimensional models. The set of feasible λs is in fact a segment in $\mathbb{R}^3$.

Define $\widehat{\gamma}_h(t)$ as the empirical correlation coefficient over h days:

$$\widehat{\gamma}_h(t) = corr(|x^f_{\cdot+h} - x^f_{\cdot}|, |x^d_{\cdot+t+h} - x^d_{\cdot+t}|).$$

where x^f and x^d are the FTSE and DJIA series of detrended log returns.

Figure 9: Distribution of log returns

(a) FTSE bulk

(b) DJIA bulk

(c) FTSE integrated tails

(d) DJIA integrated tails

Figure 10: Comparison of empirical correlations

(a) log plot; one point out of three is plotted

(b) loglog plot; for $t \geq 20$, one point out of three is plotted

Minimizing a suitable L^2 distance between this quantity and the theoretical cross-correlation (Theorem 3) we obtain

$$\lambda_1 = 0.0014; \quad \lambda_2 = 0.0005; \quad \lambda_3 = 0.$$

In Figure 11 we can see the excellent agreement of the prediction of our model and the empirical decay of the cross-asset correlations, for $t = 1, .., 400$ days.

The fact that our estimate is $\lambda_3 = 0$ means that our best fitting with real data is obtained when the shocks for FTSE are given by the shocks of the DJIA plus some additional ones,

FIGURE 11: FTSE and DJIA cross-asset correlations

(a) log plot

(b) loglog plot

given by a sparser and independent Poisson process. These estimates, due to the small sample size, are too rough to allow more quantitative considerations. In any case, if we want to see a reason for the situation above, we can suppose that shocks in the DJIA index always determine a shock in the FTSE index, whereas it is possible to see a shock in the FTSE which does not imply a significant increment in the empirical variance of DJIA.

4 Considerations on quadratic variation

The aim of this section is to provide a detailed mathematical explanation to the heuristics presented in subsection 2.3. We are going to explain in which sense V_T can be used to find shock times. This work have been developed in [4], [6], [17].

First we introduce the process $Q_T^{(N)}(t)$, which is an analogous of the estimator V_T, but its time argument is continuous. In order to simpify the following we will set $m = T - \tau_{i(T)}$ and $\alpha := \tau_{i(T)} - \tau_{i(T)-1}$. Moreover recall from subsection 2.3 that

$$Q_T(t) = \frac{I_T - I_{T-t}}{t}, \qquad \text{and } I_t = \langle X \rangle_t.$$

Definition 3. Let X_t the stochastic process defined as W_{I_t}. We define the process $Q_T^{(N)}(t)$ as the discrete version of $Q_T(t)$.

$$Q_T^{(N)}(t) := \frac{1}{t}\left(\sum_{n=0}^{N}\left(\left(X_{\frac{(n+1)T}{N}} - X_{\frac{nT}{N}}\right)^2 - \left(X_{\frac{(n+1)(T-t)}{N}} - X_{\frac{n(T-t)}{N}}\right)^2\right)\right)$$

$Q_T(\cdot)$ has some nice geometrical properties which guarantee the existence of an "isolated" maximum point in m. However we cannot observe on real data the realization of $Q_T(\cdot)$ but we can observe the process $Q_T^{(N)}(\cdot)$ on the times where it coincides with V_T. Theorem 5 shows that in suitable settings maximum points observed through $Q_T^{(N)}(\cdot)$ converges to m.

The following lemma shows geometrical properties of $Q_T(\cdot)$. Given m small enough $Q_T(\cdot)$ attains its maximum in m and the peak attained in m is arbitrarily high, i.e. reducing m increases the distance between the maximum and the next minimum.

Lemma 1. *Let m and α as in above and $K := (\frac{1}{2D})^{\frac{1}{1-2D}}$. Then*

1. m is a local maximum point for $Q_T(t)$ iff $m < K\alpha$

2. The following limit holds

$$Q_T(m) - Q_T(\alpha + m) \xrightarrow{m\to 0+} +\infty$$

Proof. 1. $Q_T(t)$ is everywhere continuous and it is differentiable but in $\{T - \tau_n\}_{n\in\mathbb{N}}$. To prove that it attains the maximum in $T - \tau_{i(T)}$ we will prove that in m the left derivative is greater than 0 and the right one is less than 0.

The derivatives are

$$Q'_T(t) = \frac{\sigma^2 2D(T - t - \tau_{i(T)-1})^{2D-1}t - (I_T - I_{T-t})}{t^2} \qquad t \in (T - \tau_{i(T)}, T - \tau_{i(T)-1})$$

$$Q'_T(t) = \frac{\sigma^2 2D(T - t - \tau_{i(T)})^{2D-1}t - (I_T - I_{T-t})}{t^2} \qquad t \in (0, T - \tau_{i(T)}) \tag{5}$$

I_s is concave then

$$I_T - I_{T-t} < I'(T-t)t$$

From (5) we get $Q'_T(t) > 0$ in $(0, m)$.

On the other hand

$$\lim_{t\to m} \frac{\sigma^2 2D(T - t - \tau_{i(T)-1})^{2D-1}t - (I_T - I_{T-t})}{t^2} = \frac{\sigma^2}{m^2}\left(2D\alpha^{2D-1}m - m^{2D}\right) =: L_\sigma(\alpha, m) \tag{6}$$

$L_\sigma(\alpha, m)$ has the following trivial properties

$$L_\sigma(\alpha, m) = 0 \Leftrightarrow m = \alpha\left(\frac{1}{2D}\right)^{\frac{1}{1-2D}}$$

$$\lim_{m\to 0^+} L_\sigma(\alpha, m) = -\infty$$

$$\lim_{m\to +\infty} L_\sigma(\alpha, m) = +\infty$$

which imply that the right derivative is less than zero if and only if $m < K\alpha$, thus $Q_T(t)$ attains a local maximum in m if and only if $m < K\alpha$.

2. Note that $Q_T \in \mathcal{C}^\infty((T - \tau_{i(T)}, T - \tau_{i(T)-1}))$ a.s. The second order derivative on this interval is the following

$$Q_T^{(2)} = \frac{2D(2D-1)(T - t - \tau_{i(T)-1})^{2D-2}}{t} - \frac{2Q'_T}{t}$$

Thus $Q'_T(t) = 0$ implies $Q_T^{(2)}(t) > 0$ then all stationary points are minimum points. Moreover Q_T can have only one minimum point which in fact exists, since from hypothesis and from (6) we get

$$\lim_{t\to T-\tau_{i(T)}} Q'_T(t) < 0$$

and

$$\lim_{t \to T - \tau_{i(T)-1}} Q'_T(t) = +\infty$$

Let $\gamma \in (T - \tau_{i(T)}, T - \tau_{i(T)-1})$ the point in which $Q_T(t)$ attains its minimum. By definition

$$Q_T(m) - Q_T(\gamma) > Q_T(m) - Q_T(\alpha + m)$$

Let $\xi = \tau_{i(T)-1} - \tau_{i(T)-2}$. We get

$$Q_T(m) - Q_T(\alpha + m) = \frac{I_T - I_{\tau_{i(T)}}}{T - \tau_{i(T)}} - \frac{I_T - I_{\tau_{i(T)-1}}}{T - \tau_{i(T)-1}} = \sigma^2 \left(\frac{m^{2D} + \alpha^{2D}}{m} - \frac{m^{2D} + \alpha^{2D} + \xi^{2D}}{m + \alpha} \right)$$

Passing to the limit

$$Q_T(m) - Q_T(\alpha + m) = \frac{\sigma^2(\alpha^{2D+1} + m^{2D}\alpha - m\xi^{2D})}{m(m + \alpha)} \xrightarrow{m \to 0+} +\infty$$

then $\lim_{m \to 0+} Q_T(m) - Q_T(\gamma) = +\infty$.

$\square$

Theorem 5. *Let $Q_T(t)$, $Q_T^{(N)}(t)$, α and m as above. Let $K := (\frac{1}{2D})^{\frac{1}{1-2D}}$ and $m < K\alpha$. Then there exists an interval I , which contains m, and the sequence $\{\mu_N\}_{N \in \mathbb{N}}$ of absolute maximum points of $Q_T^{(N)}(t)$ in I such that the following limit holds in probability*

$$\mu_N \xrightarrow{N \to \infty} m$$

In order to prove this theorem we need to apply lemma 2. The most complicated part is to define a suitable interval I such that I contains m and excludes γ - the minimum point of Q_T between m and $m + \alpha$. Obviously, given such an interval we are sure that $Q_T|_I$ is increasing before m and decreasing after m because of lemma 1. Laboriousness comes up with the fact that we are able to observe only $Q_T^{(N)}$ realization, which means that I has to be defined starting from it. To proceed with our plan we need to find the maximum point of $Q_T^{(N)}$ which corresponds (in some sense) to m: therefore we have to get rid of all the maximum points caused by the irregular realization of $Q_T^{(N)}$. The idea is to find a maximum higher than the others: the following definition moves on this direction.

Definition 4. Let $\varepsilon > 0$. We define $\mu_{N,\varepsilon}$ as the minimum $t = \frac{kT}{N}$ such that t is the absolute maximum point of $Q_T^{(N)}$ on the connected component of $Q_T^{(N)\leftarrow}(Q_T^{(N)}(t) - 2\varepsilon, +\infty)$ which contains t. We define $A_{N,\varepsilon}$ as the connected component such that $\mu_{N,\varepsilon}$ is maximum of $Q_T^{(N)}$ on $A_{N,\varepsilon}$.

Proof. Recall Theorem 6. For all $\delta > 0, \varepsilon > 0$ there exists $\bar{N}$ such that for all $N \geq \bar{N}$ the following holds

$$\mathbb{P}\left[\left\{d_\infty\left(Q_T(\cdot), Q_T^{(N)}(\cdot)\right) > \varepsilon\right\}\right] < \delta \tag{7}$$

Let $C := \{\omega \in \Omega : d_\infty(Q_T(\cdot, \omega), Q_T^{(N)}(\cdot, \omega)) > \varepsilon\}$. We will consider only $\omega \in \Omega \backslash C$ e $N \geq \bar{N}$.

Let γ be the minimum point of Q_T on the interval $(T-\tau_{i(T)}, T-\tau_{i(T)-1})$. Lemma 1 shows that for all $\varepsilon > 0$, taking T close enough to $\tau_{i(T)}$ (i.e. taking m small enough) the following inequality holds [2]:

$$Q_T(m) - Q_T(\gamma) > 4\varepsilon \tag{8}$$

Let $\mu_{N,\varepsilon}$ and $A_{N.\varepsilon}$ defined in definition 4. (8), (7) imply $\mu_{N,\varepsilon} < \gamma$, thus $\mu_{N,\varepsilon} \in (0,\gamma)$. Moreover from Lemma 1 and from the hypothesis $T-\tau_{i(T)} < K(\tau_{i(T)} - \tau_{i(T)-1})$ follows that Q_T attains the absolute maximum on the interval $(0,\gamma)$ in m. Thus

$$Q_T^{(N)}(\mu_{N,\varepsilon}) - 2\varepsilon \le Q_T(\mu_{N,\varepsilon}) - \varepsilon \le Q_T(m) - \varepsilon \le Q_T^{(N)}(m)$$

or equivalently $m \in A_{N,\varepsilon}$.

(8), definition 4 and $m \in A_{N,\varepsilon}$ implies

$$Q_T^{(N)}(\gamma) \le Q_T(\gamma) + \varepsilon < Q_T(m) - 3\varepsilon \le Q_T^{(N)}(m) - 2\varepsilon \le Q_T^{(N)}(\mu_{N,\varepsilon}) - 2\varepsilon$$

or equivalently $\gamma \notin A_{N,\varepsilon}$.

Let $\varepsilon > 0$ fixed. Consider the interval

$$I := \bigcap_{N \ge \bar{N}} A_{N,\varepsilon}$$

taking T close enough to $\tau_{i(T)}$, $m \in I$, $\gamma \notin I$. From Lemma 1 follows that $Q_T|_I$ is increasing for $x < m$ and decreasing for $x > m$. Moreover I is closed since it is intersection of closed intervals.

From Lemma 2 follows that for $\omega \in \Omega \backslash C$

$$\mu_N(\omega) \xrightarrow{N\to\infty} m(\omega)$$

Thus arbitrary choice of $\delta > 0$ in (7) implies the thesis.

□

In the proof above we have used the following results (see for instance [13] for Theorem 6).

Lemma 2. *Let I a closed interval. Let $f : I \to \mathbb{R}$ a continuous function with the following properties:*

- *f attains in m its unique local maximum in I*
- *f is strictly increasing for $x < m$, strictly decreasing for $x > m$,*
- *$\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous function in I uniformly convergent to f. Moreover, for all $n \in \mathbb{N}$, m_n is the absolute maximum point of f_n.*

Then

$$m_n \xrightarrow{N\to\infty} m$$

Theorem 6. *Let $M \in \mathcal{M}^{c,loc}$. The process $S_n^{(2)}(M)$ converges to $\langle M\rangle$ in probability, uniformly on compact intervals $[0,T]$.*

[2] Using the same argument of the proof of Lemma 1 we see that there is only a minimum point between two maximum points, thus $Q_T(m) - Q_T(\gamma) \xrightarrow{m\to 0+} +\infty$

5 Proofs of results on the bivariate model

5.1 Proof of theorem 4

Proof. We start the computations on $Cov(\xi_s, \eta_t)$ writing more explicitly the quantities involved. Recall that the increments of W^X and W^Y are independent on disjoint time intervals, and W^X and $\tilde{W}$ are independent Brownian Motions. So for $h < t - s$

$$\begin{aligned}
Cov(\xi_s, \eta_t) &= \mathbb{E}(|X_{s+h} - X_s||Y_{t+h} - Y_t|) - \mathbb{E}|X_{s+h} - X_s|\mathbb{E}|Y_{t+h} - Y_t| \\
&= \mathbb{E}\left(|W_1^X|\sqrt{I_{s+h}^X - I_s^X}|\tilde{W}_1|\sqrt{I_{t+h}^Y - I_t^Y}\right) \\
&\quad - \mathbb{E}\left(|W_1^X|\sqrt{I_{s+h}^X - I_s^X}\right)\mathbb{E}\left(|\tilde{W}_1|\sqrt{I_{t+h}^Y - I_t^Y}\right)
\end{aligned}$$

and using independence

$$\begin{aligned}
Cov(\xi_s, \eta_t) &= (\mathbb{E}|W_1^X|)^2 Cov\left(\sqrt{I_{s+h}^X - I_s^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right) \\
&= \frac{2}{\pi} Cov\left(\sqrt{I_{s+h}^X - I_s^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right).
\end{aligned}$$

From our choice of $\mathcal{T}^X$ and $\mathcal{T}^Y$ we have the stationarity of the increments of (I^X, I^Y), therefore

$$Cov\left(\sqrt{I_{s+h}^X - I_s^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right) = Cov\left(\sqrt{I_h^X}, \sqrt{I_{t-s+h}^Y - I_{t-s}^Y}\right).$$

Remark that the covariance of the absolute values of the returns actually depends just on $Cov\left(\sqrt{I_h^X}, \sqrt{I_{t+h}^Y - I_t^Y}\right)$, where t is the time difference. Recall

$$I_h = \sigma^2 \left[(h - \tau_{i(h)})^{2D} + \sum_{k=1}^{i(h)} (\tau_k - \tau_{k-1})^{2D} - (-\tau_0)^{2D}\right]$$

Almost surely, for h small enough, $i(h) = i(0) = 0$, so the sum in the right hand vanishes and a.s.

$$\begin{aligned}
\lim_{h \downarrow 0} \frac{I_h}{h} &= \lim_{h \downarrow 0} \sigma^2 \frac{(h - \tau_{i(h)})^{2D} - (-\tau_0)^{2D}}{h} \\
&= \sigma^2 \lim_{h \downarrow 0} \frac{(h - \tau_0)^{2D} - (-\tau_0)^{2D}}{h} = 2D\sigma^2(-\tau_0)^{2D-1},
\end{aligned}$$

and analogously

$$\lim_{h \downarrow 0} \frac{I_{t+h} - I_t}{h} = 2D\sigma^2(t - \tau_{i(t)})^{2D-1}.$$

Lemma 3 implies the uniform integrability of the families

$$\left\{\frac{I_h^X}{h} : h \in (0, 1]\right\}, \quad \left\{\frac{I_{t+h}^Y - I_t^Y}{h} : h \in (0, 1]\right\},$$

therefore we first apply bi-linearity of covariance and then take the limit inside, obtaining

$$\lim_{h\downarrow 0}\frac{Cov\left(\sqrt{I_h^X},\sqrt{I_{t+h}^Y-I_t^Y}\right)}{h}=Cov\left(\lim_{h\downarrow 0}\sqrt{\frac{I_h^X}{h}},\lim_{h\downarrow 0}\sqrt{\frac{I_{t+h}^Y-I_t^Y}{h}}\right)$$
$$=2\sqrt{D^X D^Y}\sigma^X\sigma^Y Cov\left((-\tau_0^X)^{D^X-1/2},(t-\tau_{i^Y(t)}^Y)^{D^Y-1/2}\right).$$

We can obtain a better representation of this quantity multiplying the right term in the covariance by the characteristic function of $\{i^Y(t)=0\}$ plus the characteristic function of its complement:

$$\begin{aligned}
&Cov\left((-\tau_0^X)^{D^X-1/2},(t-\tau_{i^Y(t)}^Y)^{D^Y-1/2}\right)\\
&\quad=Cov\left((-\tau_0^X)^{D^X-1/2},(t-\tau_{i^Y(t)}^Y)^{D^Y-1/2}\mathbf{1}_{\{i^Y(t)=0\}}\right)\\
&\quad+Cov\left((-\tau_0^X)^{D^X-1/2},(t-\tau_{i^Y(t)}^Y)^{D^Y-1/2}\mathbf{1}_{\{i^Y(t)>0\}}\right).
\end{aligned}$$

The second covariance is 0 because $(t-\tau_{i^Y(t)}^Y)^{D^Y-1/2}\mathbf{1}_{\{i^Y(t)>0\}}$ is $\mathcal{G}_{>0}^Y$ measurable, where $\mathcal{G}_{>0}^Y=\sigma(\tau_k^Y : k>0)$, and $\mathcal{G}_{>0}^Y$ is independent of τ_0 (loss of memory property of Poisson processes). So, using the fact that $\mathbf{1}_{\{i^Y(t)=0\}}$ is $\mathcal{G}_{>0}^Y$ measurable, because so is $\mathbf{1}_{\{i^Y(t)>0\}}$, we have

$$\begin{aligned}
&Cov\left((-\tau_0^X)^{D^X-1/2},(t-\tau_{i^Y(t)}^Y)^{D^Y-1/2}\right)\\
&\quad=Cov\left((-\tau_0^X)^{D^X-1/2},(t-\tau_0^Y)^{D^Y-1/2}\mathbf{1}_{\{i^Y(t)=0\}}\right)\\
&\quad=Cov\left((-\tau_0^X)^{D^X-1/2},(t-\tau_0^Y)^{D^Y-1/2}\right)E\left(\mathbf{1}_{\{i^Y(t)=0\}}\right)\\
&\quad=Cov\left((-\tau_0^X)^{D^X-1/2},(t-\tau_0^Y)^{D^Y-1/2}\right)e^{-\lambda^Y t},
\end{aligned}$$

and the theorem is proved. □

5.2 Uniform integrability of the time change

We present now the technical lemma used in the proof of Theorem 4. Recall that $0<D<1/2$.

Lemma 3. *The class of random variables*

$$\left\{\frac{I_h^X}{h} : h\in(0,1]\right\}$$

is bounded in L^δ for $\delta<\frac{1}{1-2D}$.

Proof. Recall

$$I_t=\sigma^2\left[(t-\tau_{i(t)})^{2D}+\sum_{k=1}^{i(t)}(\tau_k-\tau_{k-1})^{2D}-(-\tau_0)^{2D}\right]$$

and decompose $\mathbb{E}(I_t^\delta)$

$$\mathbb{E}(I_t^\delta) = \mathbb{E}(I_t^\delta | i(t) = 0)\mathbb{P}(i(t) = 0) + \sum_{k=1}^{\infty} \mathbb{E}(I_t^\delta | i(t) = k)\mathbb{P}(i(t) = k)$$

Conditioning on $i(t) = 0$ and using convexity,

$$I_t = \sigma^2 \left[(t - \tau_0)^{2D} - (-\tau_0)^{2D} \right] \leq 2D\sigma^2(-\tau_0)^{2D-1} t$$

in a right neighborhood of $t = 0$. So

$$\mathbb{E}(I_t^\delta | i(t) = 0) \leq (2D)^\delta \sigma^{2\delta} \mathbb{E}\left((-\tau_0)^{\delta(2D-1)} \right) t^\delta \leq C_0 t^\delta$$

for $\delta < \frac{1}{1-2D}$, since $-\tau_0$ is an random variable with exponential distribution. Conditioning on $i(t) = k$, $k \geq 1$, and using convexity again,

$$\begin{aligned}
I_t &= \sigma^2 \left[(t - \tau_k)^{2D} + \sum_{j=1}^{k} (\tau_j - \tau_{j-1})^{2D} - (-\tau_0)^{2D} \right] \\
&\leq \sigma^2 \left[(t - \tau_k)^{2D} + \sum_{j=2}^{k} (\tau_j - \tau_{j-1})^{2D} + (t - \tau_0)^{2D} - (-\tau_0)^{2D} \right] \\
&\leq \sigma^2 \left[(t - \tau_k)^{2D} + \sum_{j=2}^{k} (\tau_j - \tau_{j-1})^{2D} + 2D(-\tau_0)^{2D-1} t \right].
\end{aligned}$$

By Jensen inequality and the fact that $2D < 1$,

$$(t - \tau_k)^{2D} + \sum_{j=2}^{k} (\tau_j - \tau_{j-1})^{2D} \leq k \left(\frac{(t - \tau_k) + \sum_{j=2}^{k} (\tau_j - \tau_{j-1})}{k} \right)^{2D} \leq k \left(\frac{t}{k} \right)^{2D}$$

Then

$$I_t \leq \sigma^2 \left(2D(-\tau_0)^{2D-1} t + k \left(\frac{t}{k} \right)^{2D} \right).$$

Now, supposing $t \leq 1$, we have that for suitable positive constants C_1 and C_2

$$\mathbb{E}(I_t^\delta | i(t) = k)\mathbb{P}(i(t) = k) \leq C_1 \frac{\lambda^k}{k!} t^\delta + C_2 k^{\delta(1-2D)} \frac{\lambda^k}{k!} t^{1+2D\delta}.$$

Recall $\delta < \frac{1}{1-2D}$. Therefore $\delta < 1 + 2D\delta$, so $t^{1+2D\delta} \leq t^\delta$, and then

$$\mathbb{E}(I_t^\delta | i(t) = k)\mathbb{P}(i(t) = k) \leq \left(C_1 + C_2 k^{\delta(1-2D)} \right) \frac{\lambda^k}{k!} t^\delta.$$

Therefore

$$\mathbb{E}(I_t^\delta) \leq \left[C_0 + \sum_{k=1}^{\infty} C_3 \frac{\lambda^k}{k!} \right] t^\delta \leq C_4 t^\delta$$

where C_3 and C_4 are positive constants. So $\left\{ \frac{I_t^X}{t} : t \in (0, 1] \right\}$ is bounded in L^δ. □

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Mario Bonino
Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Padova, via Trieste 63, I-35121 Padova, Italy.
Email: `mario.bonino@outlook.com`

Matteo Camelia
Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Padova, via Trieste 63, I-35121 Padova, Italy.
Email: `matteo.camelia@studenti.unipd.it`

Paolo Pigato
Dipartimento di Matematica Pura ed Applicata, Università degli Studi di Padova, via Trieste 63, I-35121 Padova, Italy.
Laboratoire d'Analyse et de Mathematiques Appliquées, UMR 8050, Université Paris-Est Marne-la-Vallée, 5 Bld Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée Cedex 2, France.
Email: `pigato@math.unipd.it`

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